Chapter 10: Application to Optimal Stopping

10.1. In each of the optimal stopping problems below, find the supremum g^* and—if it exists—an optimal stopping time τ^*. (Here B_t denotes 1-dimensional Brownian motion.)

(a) $g^*(x) = \sup_\tau E^x(B_\tau^2)$
$g^*(x) = \sup_\tau E^x(B_\tau^2) = \sup_\tau \tau = \infty$, so τ^* does not exist

(b) $g^*(x) = \sup_\tau E^x(|B_\tau|^p)$ for $p > 0$
$g^*(x) = \infty$ so again, τ^* does not exist

(c) $g^*(x) = \sup_\tau E^x(e^{-B_\tau^2}) = 1$ since when $B_\tau^2 = 0$, this is maximized. So τ^* exists. It is the first time we see that $B_t = 0$, $\tau^* = \inf\{t > 0 \mid B_t = 0\}$. If we defined Brownian motion as $B_0 = 0$, stop immediately.

(d) $g^*(x) = \sup_\tau E^{(s,x)}(e^{-\rho(s+\tau)}) \cosh B_\tau)$
$\cosh x = \frac{1}{2}(e^x + e^{-x})$, so $g^*(x) = \sup_\tau E^{(s,x)}(e^{-\rho(s+\tau)})\frac{1}{2}(e^{B_\tau} + e^{-B_\tau}))$
$= \sup_\tau E^{(s,x)}(\frac{1}{2}e^{B_\tau - \rho(s+\tau)} + \frac{1}{2}e^{-\rho(s+t) - B_\tau})$
$= g(s,x)$ if $\rho \geq 1/2$, and ∞ if $\rho < 1/2$. In the former case, $\tau^* = 0$ and in the latter case, it doesn't exist.