Chapter 12: Applications to Mathematical Finance

Exercise Solutions: Exercises 12.1, 12.2, 12.6(a)

12.1 Prove that the price process $\{X_t\}_{t\in[0,T]}$ has an arbitrage iff the normalized price process $\{\overline{X_t}\}_{t\in[0,T]}$ has an arbitrage.

We want to confirm that normalization doesn't mess anything up.
$\Longrightarrow$: Let θ_t be an arbitrage in $\{X_t\}$. Then we have that

1. θ is admissable in $\{X_t\}$. So value process lower bound V_t^θ exists, which means that $\overline{V_t^\theta}$ also exists, since we produce it by multiplying V_t^θ by a constant.
2. θ is self-financing in $\{\overline{X_t}\}$
3. θ is s.t. $\overline{V^\theta}(T) \geq 0$ almost surely and $P(\overline{V^\theta}(T) > 0) > 0$. This is because, if $V_t^\theta > 0$, then $\overline{V}_t^\theta > 0$ since they only differ by a positive multiplicative constant.

$\Longleftarrow$: The same reasoning works for the other direction.

12.2. Let $\theta(t) = (\theta_0, \ldots, \theta_n)$ be a constant portfolio. Prove that θ is self-financing.

The first condition is satisfied, since θ is constant. We want to show that $dV(t) = \theta(t) \cdot dX(t)$. This is true because $V(t) = \sum_{i=0}^n \theta_i X_i(t) = \int_0^t \theta(s) dX(s)$.

12.6. Determine if the following normalized markets $\{X_t\}_{t\in[0,T]}$ allow an arbitrage. If so, find one.

(a) $dX_1(t) = 3dt + dB_1(t) + dB_2(t)$
$dX_2(t) = -dt + dB_1(t) - dB_2(t)$
We can construct an equivalent martingale measure for this market, so there is no arbitrage.