Chapter 5: Stochastic Differential Equations

Exercise Solutions: Exercises 5.1, 5.3, 5.4

5.1. Verify that the given processes solve the given corresponding stochastic differential equations (B_t denotes 1-dimensional Brownian motion)

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$: Apply Itô's formula on $g(x,t) = e^x$ to get that
$dX_t = e^x dB_t + \frac{1}{2}e^x(dB_t)^2 = X_t dB_t + \frac{1}{2}X_t dt$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1} \end{pmatrix} dB_t$

Do this separately. $dX_t = dt$ is solved by $X_t(t) = t$, since by Itô's formula with $g(x,t) = t$, $dX_t = dt$ (everything else 0).
We want to verify that $dX_2 = X_2 dt + e^{X_1} dB_t = X_2 dt + e^t dB_t$ is solved by $X_2 = e^t B_t$. Use Itô's formula with $g(x,t) = e^t x$. Then $dX_2 = e^t x dt + e^t dB_t = X_2 dt + e^t dB_t$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} dB_t$

$dX_1 = \frac{1}{2}X_1 dt + X_2 dB_t$: Use Itô's formula with $g(x,t) = \cosh(x)$ to get $dX_1 = \sinh(x) + \frac{1}{2}\cosh(x)dt$
$dX_2 = \frac{1}{2}X_2 dt + X_1 dB_t$: Use Itô's formula with $g(x,t) = \sinh(x)$ to get $dX_2 = \cosh(x)dB_t + \frac{1}{2}\sinh(x)dt$

5.3. Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbb{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation $dX_t = rX_t dt + X_t(\sum_{k=1}^n \alpha_k dB_k(t))$ for $X_0 > 0$. (This is a model for exponential growth and sevearl independent white noise sources in the relative growth rate.)

This is almost identical to Example 5.1.1:
$\frac{1}{X_t} dX_t = rdt + \sum_{k=1}^n \alpha_k dB_k(t)$ $(\star)$
$\int_0^t \frac{1}{X_t} dX_t = rt + \sum_{k=1}^n \alpha_k B_k(t)$
$d(\log X_t) = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2}(dX_t)^2$ by Itô's formula with $g(t,x) = \log x$
$d(\log X_t) = \frac{1}{X_t} dX_t - \frac{1}{2}\sum_{k=1}^n \alpha_k^2 dt$ by $(dX_t)^2 = (rX_t dt + X_t \sum_{k=1}^n \alpha_k dB_k(t))^2 = X_t^2 \sum_{k=1}^n \alpha_k^2 dt$
$\frac{1}{X_t} dX_t = d(\log X_t) + \frac{1}{2}\sum_{k=1}^n \alpha_k^2 dt$ $(\star\star)$

Then we have $rdt + \sum_{k=1}^n \alpha_k dB_k(t) = d(\log X_t) + \frac{1}{2}\sum_{k=1}^n \alpha_k^2 dt$
$d(\log X_t) = rdt + \sum_{k=1}^n \alpha_k dB_k(t) - \frac{1}{2}\sum_{k=1}^n \alpha_k^2 dt$
$= (r - \frac{1}{2}\sum_{k=1}^n \alpha_k^2)dt + \sum_{k=1}^n \alpha_k dB_k(t)$

So like with separable ODEs, we write
$\int_0^t d(\log X_t) = \int_0^t (r - \frac{1}{2}\sum_{k=1}^n \alpha_k^2)dt + \int_0^t \sum_{k=1}^n \alpha_k dB_k(t)$
$\log X_t = \log X_0 + t(r - \frac{1}{2}\sum_{k=1}^n \alpha_k^2) + \sum_{k=1}^n \alpha_k B_k(t)$
$X_t = X_0 \exp(t(r - \frac{1}{2}\sum_{k=1}^n \alpha_k^2) + \sum_{k=1}^n \alpha_k B_k(t))$

5.4. Solve the following stochastic differential equations:

(ii) $dX_t = X_t dt + dB_t$. Hint: Multiply both sides with 'the integrating factor' e^{-t} and compare with $d(e^{-t}X_t)$

Ww follow the hint: $e^{-t}dX_t = e^{-t}X_t dt + e^{-t}dB_t$. We want to compare this with $d(e^{-t}X_t)$, so we use Itô's formula on $g(x,t) = e^{-t}x$, which, to our pleasant surprise, yields
$d(e^{-t}X_t) = -X_t e^{-t}dt + e^{-t}dX_t = -e^{-t}X_t dt + e^{-t}dX_t$

Then $e^{-t}dB_t = -e^{-t}X_t dt + e^{-t}dX_t = d(e^{-t}X_t)$
$\int_0^t e^{-t}dB)t = \int_0^t d(e^{-t}X_t)$
$\int_0^t e^{-t}dB_t = e^{-t}X_t$
$X_t = e^t \int_0^t e^{-s}dB_s + X_0 e^t$