Chapter 7 - Diffusions: Basic Properties

Exercise Solutions: Exercises 7.1, 7.2, 7.12

7.1. Find the generator of the following Itô diffusions:

First, we state Theorem 7.3.3 which will come of great aid: If X_t is the Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, and $f \in C_0^2(\mathbb{R}^n)$, then $f \in \mathcal{D}_A$ and $Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$

(a) Ornstein-Uhlenbeck process: $dX_t = \mu X_t dt + \sigma dB_t$ for $B_t \in \mathbb{R}$, μ, σ constants

$\sum_i b_i(x)\frac{partial f}{\partial x_i} = \mu x \frac{\partial f}{\partial x_i}$ and $\sum_{i,j}(\sigma\sigma^T)(x) = \sigma^2$
So $Af(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 f''(x)$ if $f \in C_0^2(\mathbb{R}^n)$

(b) Geometric Brownian motion: $dX_t = rX_t dt + \alpha X_t dB_t$ for $B_t \in \mathbb{R}$, r, α constants

$\sum_i b_i(x)\frac{\partial f}{\partial x_i} = rxf'(x)$ and $\sum_{i,j}(\sigma\sigma^T)(x) = \alpha^2 x^2$
So $Af(x) = rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x)$ if $f \in C_0^2(\mathbb{R}^n)$

(c) $dY_t = rdt + \alpha Y_t dB_t$ for $B_t \in \mathbb{R}$, r, α constants

$\sum_i b_i(y)\frac{\partial f}{\partial y_i} = rf'(y)$ and $\sum_{i,j}(\sigma\sigma^T)(y) = \alpha^2 y^2$
So $Af(x) = rf'(y) + \frac{1}{2}\alpha^2 y^2 f''(y)$ if $f \in C_0^2(\mathbb{R}^n)$

(e) $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_1} \end{pmatrix} dB_t$ for $B_t \in \mathbb{R}$

$\sum_i b_i(x)\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2}$ and $\sum_{i,j}(\sigma\sigma^T)(x)\frac{\partial^2 f}{\partial x_i^2} = 0 + \frac{1}{2}e^{2x_1}\frac{\partial^2 f}{\partial x_2^2}$
So $Af(x) = \frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \frac{1}{2}e^{2x_1}\frac{\partial^2 f}{\partial x_2^2}$

(f) $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$

$\sum_i b_i(x)\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_1} + 0$ and $\sum_{i,j}(\sigma\sigma^T)(x)\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial x_1^2} + x_1^2\frac{\partial^2 f}{\partial x_2^2}$
So $Af(x) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}x_1^2\frac{\partial^2 f}{\partial x_2^2}$

(g) $X(t) = (X_1, \ldots, X_n)$ where $dX_k(t) = r_k X_k dt + X_k \cdot \sum_{j=1}^n \alpha_{kj} dB_j$ for $1 \le k \le n$, $(B_1, \ldots, B_n)$ Brownian motion in $\mathbb{R}^n$, r_k, α_{kj} constants

$\sum_i b_i(x)\frac{\partial f}{\partial x_i} = \sum_k r_k x_k \frac{\partial f}{\partial x_k}$ and $\sum_{i,j}(\sigma\sigma^T)(x)\frac{\partial^2 f}{\partial x_i^2} = \sum_{i,j} x_i x_j \sum_k \alpha_{ik}\alpha_{jk}\frac{\partial^2 f}{\delta x_i \delta x_j}$
So $Af(x) = \sum_k r_k x_k \frac{\partial f}{\partial x_k} + \frac{1}{2}\sum_{i,j} x_i x_j (\sum_k \alpha_{ik}\alpha_{jk})\frac{\partial^2 f}{\partial x_i \partial x_j}$

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7.2 Find an Itô diffusion (i.e: write down the stochastic differential equation for it) whose generator is the following:

(b) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}$ for $f \in C_0^2(\mathbb{R}^2)$

We work backwards from Theorem 7.3.3. Guess that b is something like cX, since then we'd get $cx\frac{\partial f}{\partial x}$ in the generator. Guess that σ is something like αx, again so that we'd get $\frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x_2}$ in the generator. Since we have 3 terms, from 7.1 (f) we guess this is something 2-dimensional.

This works: $\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ cX_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha X_2 \end{pmatrix} dB_t$. Apply Theorem 7.3.3 to get the desired generator. Maybe this isn't unique—hope to find out someday

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7.12 (Local martingales.) An $\mathcal{N}_t$-adapted stochastic process $Z_t \in \mathbb{R}^n$ is called a local martingale with respect to the given filtration $\{\mathcal{N}_t\}$ if there exists an increasing sequence of $\mathcal{N}_t$-stopping times τ_k s.t. $\tau_k \to \infty$ almost surely as $k \to \infty$, and $Z(t \wedge \tau_k)$ is an $\mathcal{N}_t$-martingale for all k.

(a) Show that if Z_t is a local martingale and there exists a constant $T \le \infty$ s.t. that the family $\{Z_t\}_{t \le T}$ is uniformly-integrable, then $\{Z_t\}_{t \le T}$ is a martingale.

We want to show that $E(Z_t \mid \mathcal{F}_s) = Z_s$ for $z < t \le T$. Since Z_t is a local martingale, we have
$E(Z_{t\wedge\tau_k} \mid \mathcal{F}_s) = Z_{s\wedge\tau_k}$
As $k \to \infty$, $Z_{s\wedge\tau_k} \to Z_s$, $Z_{t\wedge\tau_k} \to Z_t$. By uniform integrability, $E(Z_t \mid \mathcal{F}_s) = \lim_{k\to\infty} E(Z_{t\wedge\tau_k} \mid \mathcal{F}_s) = \lim_{k\to\infty} Z_{s\wedge\tau_k} = Z_s$

(b) In particular, if Z_t is a local martingale and there exists a constant $K < \infty$ s.t. $E(Z_\tau^2) \le K$ for all $\tau \le T$, then $\{Z_t\}_{t \le T}$ is a martingale.

The condition $E(Z_\tau^2) \le K$ for all $\tau \le T$ means that $\{Z_\tau\}_{\tau \le T}$ is uniformly integrable, so we apply part (a).

(c) Show that, if Z_t is a lower bounded local martingale, then Z_t is a supermartingale.

We want to show that Z_t. is s.t. $E(Z_t \mid \mathcal{F}_s) \le Z_s$. Use Fatou's lemma from Chapter 5 to get that
$E(Z_t \mid \mathcal{F}_s) \le \lim_{k\to\infty} E(Z_{t\wedge\tau_k} \mid \mathcal{F}_s)$, which is Z_s by Z_t being a local martingale.