Chapter 3: The Itô Integral

Notes | Exercise Solutions

Notes:

- We want to know if it is sensible for us to define the following object: $X_T = X_S + \int_S^T b(t, X_t)dt + \int_S^T \sigma(t, X_t)dB_t$. Does it make sense to write $\int_S^T \sigma(t, X_t)dB_t$?
- It makes sense under certain constructions. The third term $\int_S^T \sigma(t, X_t)dB_t$ is the Itô integral, and we will construct it in a similar way as we constructed Riemann integrals.
  1. For any $f(t,\omega)$, assume that it has the form $f(t,\omega) = \sum_{j\geq 0} e_j(\omega) \cdot \chi_{[\frac{j}{2^n}, \frac{j+1}{2^n})}(t)$ where χ is the indicator function that's 1 if $t \in [\frac{j}{2^n}, \frac{j+1}{2^n})$ and 0 otherwise
  2. Approximate f by $\sum_j f(t_j^*, \omega) \cdot \chi_{[t_j, t_{j+1})}(t)$ and define $\int_S^T \sigma(t, X_t)dB_t$ as follows: Take f to be Brownian motion increments, adn take $t_j^* = t_j$ (that is, the left end point). Unlike Riemann integrals, the choice of end point matters. Choosing the middle end point yields the Stratonovich integral, and choosing the right end point yields the Hänggi-Klimontovich/anti-Itô (lol) integral. Picking the left end point to yield the Itô integral makes sense for our purposes here because we'd like to view t as time, and f at time t_0 ideally shouldn't depend on some future time t_1.
- Def ($\mathcal{V}$): $\mathcal{V} = \mathcal{V}(S,T)$ is the class of functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ s.t.
  1. Measurable: $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $[0,\infty)$
  2. Adapted: $f(t,\omega)$ is adapted to $\mathcal{F}_t$
  3. Finite variance: $E[\int_S^T f(t,\omega)^2 dt] < \infty$
- Properties of the Itô integral:
  - Thm (Ito's isometry) Let $\phi(t,\omega)$ be bounded and elementary, Then $E(\int_S^T \phi(t,\omega)dB_t(\omega))^2) = E(\int_S^T \phi(t,\omega)^2 dt)$. As a corollary, if $f \in \mathcal{V}(S,T)$ then $E((\int_S^T f(t,\omega)dB_t)^2) = E(\int_S^T f^2(t,\omega)dt)$. This helps us compute variance.
  - Applying this to Brownian motion with $B_0 = 0$ yields $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$
  - Thm (Operations) Let $f, g \in \mathcal{V}(0,T)$ and $0 \leq S < U < T$. Then
    1. Sum rule: $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for almost all ω
    2. Scalar multiplication: $\int_S^T c f dB_t = c\int_S^T f dB_t$ for all $c \in \mathbb{R}$, almost all ω
    3. Expectation: $E(\int_S^T f dB_t) = 0$
    4. Measurability: $\int_S^T f dB_t$ is $\mathcal{F}_T$-measurable

Exercise Solutions

Exercises 3.4, 3.6

3.4. Check whether the following processes are martingales wrt $\{\mathcal{F}_t\}$:

First, recall that $\{X_t\}$ on $(\Omega, \mathcal{F}, P)$ is a martingale wrt $\{\mathcal{F}_t\}$ iff

1. X_t is $\mathcal{F}_t$-measurable for all t,
2. $E(|X_t|) < \infty$ for all t, and
3. $E(X_s \mid \mathcal{F}_t) = X_t$ for all $s \geq t$.

(i) $X_t = B_t + 4t$

1. satisfied, because B_t is $\mathcal{F}_t$-measurable
2. satisfied, because $E(|X_t|) = E(|B_t + 4t|) \leq E(|B_t|) + E(|4t|) = E(|B_t|) + 4t < \infty$
3. not satisfied: $E(X_s \mid \mathcal{F}_t) = E(B_s + 4s \mid \mathcal{F}_t) = E(B_s \mid \mathcal{F}_t) + E(4s \mid \mathcal{F}_t) = B_t + 4s$ since $s \geq t$

So X_t is not a martingale wrt $\{\mathcal{F}_t\}$.

(ii) $X_t = B_t^2$

1. satisfied, since B_t is $\mathcal{F}_t$-measurable
2. satisfied, since $E(B_t^2) = (E(B_t))^2 = t < \infty$
3. not satisfied: $E(X_s \mid \mathcal{F}_t) = E(B_s^2 \mid \mathcal{F}_t) = E_t((B_s - B_t)^2 + 2B_sB_t - B_t^2)$, since $(B_s - B_t)^2 = B_s^2 - 2B_sB_t + B_t^2$. (This is a common trick with martingales.) Continuing, $E(X_s \mid F_t) = E_t((B_s - B_t)^2) + E_t(2B_sB_t) - E_t(B_t^2) = s - t + 2E_t(B_s)E_t(B_t) - B_t^2$ by independence $= s - t - X_t \neq X_t$

So X_t is not a martingale wrt $\{\mathcal{F}_t\}$.

3.6. Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

1. B_t is $\mathcal{F}_t$-measurable
2. $E(|N_t|) = E(|B_t^3 - 3tB_t|) \leq E(|B_t^3|) + E(|3tB_t|) = E(|B_t^3|) + 3tE(|B_t|) < \infty$
3. $E_t(N_s \mid) = E_t(B_s^3 - 3sB_s)$
 $= E_t((B_t + (B_s - B_t))^3) - 3sE_t(B_t + (B_s - B_t))$ by $B_s = B_t + (B_s - B_t)$ (this is again a common trick with martingales)
 $= E_t(B_t^3) + E_t(B_t(B_s - B_t)^2) + E_t(B_t^2(B_s - B_t)) + \cancel{E_t((B_s - B_t)^3)} - 3sE_t(B_t) + \cancel{3sE_t(B_s - B_t))}$
 $= B_t^3 + 3B_tE_t((B_s - B_t^2)) + \cancel{3B_t^2E_t(B_s - B_t)} - 3sB_t$
 $= B_t^3 + 3B_t(s - t) - 3sB_t$
 $= B_t^3 + 3sB_t - 3tB_t - 3sB_t$
 $= B_t^3 - 3tB_t = N_t$

So N_t is a martingale wrt $\{\mathcal{F}_t\}$.