Chapter 4: The Itô Formula and the Martingale Representation Theorem

Notes | Exercise Solutions

Notes:

- Def (1-dimensional Itô process): If B_t is a 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, then say that a stochastic process X_t is a 1-dimensional Itô process on the same probability space if it is given by $X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$ where $v \in \mathcal{W}_{\mathcal{H}}$. Then we can express X_t in the following way: $dX_t = udt + vdB_t$.
- Thm (1-dimensional Itô formula): For $dX_t = udt + vdB_t$, define $Y_t = g(t, X_t)$ for g twice continuously differentiable $[0,\infty) \times \mathbb{R}$. Then $dY_t = \frac{\delta g}{\delta t}(t, X_t)dt + \frac{\delta g}{\delta x}(t, X_t)dX_t + \frac{1}{2}\frac{\delta^2 g}{\delta x^2}(t, X_t) \cdot (dX_t)^2$ where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed using the following rules:
  - $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$
  - $dB_t \cdot dB_t = dt$
- Thm (Integration by parts): Let $f(s,\omega)$ be continuous and of bounded variation wrt $s \in [0,t]$ for almost all ω . Then $\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s$
  - Thm (Martingale representation): Let $B(t) = (B_1(t), \ldots, B_n(t))$. Let M_t be an $\mathcal{F}_t^{(n)}$-martingale wrt P and $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s,\omega)$ s.t. $g \in \mathcal{V}^{(n)}(0,t)$ for all $t \geq 0$ and $M_t(\omega) = E(M_0) + \int_0^t g(s,\omega)dB(s)$ almost surely for all $t \geq 0$

Exercise Solutions:

Exercises 4.1 a) 4.2

4.1 a) Use Itô's formula to write X_t in the standard form $dX_t = u(t,\omega)dt + v(t,\omega)dB_t$ for $X_t = B_t^2$ where B_t is 1-dimensional:

Take $g(x,t) = x^2$ and $X_t = B_t^2$. Then $\frac{\delta g}{\delta t}(t, X_t) = 0$, $\frac{\delta g}{\delta x}(t, X_t) = 2x$, and $\frac{\delta^2 g}{\delta x^2}(t, X_t) = 2$.
So $dB_t^2 = 2B_t dB_t + (dB_t)^2 = 2B_t dB_t + dt$

4.2 Use Itô's formula to prove that $\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$

Take $g(x,t) = \frac{1}{3}x^3$, $X_s = B_s$. Then by Itô's formula, we have that $dY_s = d(\frac{1}{3}B_s^3) = B_s^2 dB_s + B_s ds$, so $\frac{1}{3}B_t^3 = \int_0^t B_s^2 dB_s + \int_0^t B_s ds$. Rearranging, this is $\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$